

# Cost and Effects of Pinning Control for Network Synchronization \*

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**Abstract.** In this paper, the problem of pinning control for synchronization of complex dynamical networks is discussed. A cost function of the controlled network is defined by the feedback gain and the coupling strength of the network. An interesting result is that lower cost is achieved by the control scheme of pinning nodes with smaller degrees. Some rigorous mathematical analysis is presented for achieving lower cost in the synchronization of different star-shaped networks. Numerical simulations on some non-regular complex networks generated by the Barabási-Albert model and various star-shaped networks are shown for verification and illustration.

**Keywords:** complex dynamical network, pinning control, exponential stability.

## 1 Introduction and problem formulation

Complex networks are currently being studied across many fields of sciences, including physics, chemistry, biology, mathematics, sociology and engineering [1, 2, 3, 5, 9, 15, 17, 19]. A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. Examples of complex networks include the Internet, food webs, cellular neural networks, biological neural networks, electrical power grids, telephone cell graphs, etc. Recently, synchronization of complex networks of dynamical systems has received a great deal of attention from the nonlinear dynamics community [10, 12, 16, 18, 20]. A special control strategy called *pinning control* is used to achieve synchronization of complex networks; that is, only a fraction of the nodes or even a single node is controlled over the whole network [4, 6, 11, 21]. This control method has become a common technique for control, stabilization and synchronization of coupled dynamical systems. In general, different nodes have different degrees in a network, thus a natural question is how different the effect would be when nodes with different degrees are pinned.

Consider a dynamical network consisting of  $N$  identical and diffusively coupled nodes, with each node being an  $n$ -dimensional dynamical system. The state equations of the network are

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t), \quad i = 1, 2, \dots, N, \quad (1)$$

where  $f(\cdot)$  is the dynamical function of an isolated node,  $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in R^n$  are the state variables of node  $i$ , constant  $c > 0$  represents the coupling strength, and  $\Gamma \in R^{n \times n}$  is the

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inner linking matrix. Moreover, the coupling matrix  $A = (a_{ij}) \in R^{N \times N}$  represents the coupling configuration of the network: If there is a connection between node  $i$  and node  $j$  ( $i \neq j$ ), then  $a_{ij} = a_{ji} = 1$ ; otherwise,  $a_{ij} = a_{ji} = 0$  ( $i \neq j$ ); the diagonal entries of  $A$  are defined by

$$a_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij}, \quad i = 1, 2, \dots, N. \quad (2)$$

Suppose that the network is connected in the sense of having no isolated clusters. Then, the coupling matrix  $A$  is irreducible. From Lemma 2 of [20], it can be proved that zero is an eigenvalue of  $A$  with multiplicity one and all the other eigenvalues of  $A$  are strictly negative.

Network (1) is said to achieve (asymptotical) synchronization if

$$x_1(t) \rightarrow x_2(t) \rightarrow \dots \rightarrow x_N(t) \rightarrow s(t), \quad \text{as } t \rightarrow \infty, \quad (3)$$

where, because of the diffusive coupling configuration,  $s(t)$  is a solution of an isolated node, which can be an equilibrium, a periodic or a chaotic orbit. As shown in [4, 6, 11, 21], this can be achieved by controlling several nodes (or even only one node) of the network. Without loss of generality, suppose that the controllers are added on the last  $N - k$  nodes of the network, so that the equations of the controlled network can be written as

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t), \quad i = 1, 2, \dots, k, \quad (4)$$

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t) - c \varepsilon_i \Gamma (x_i(t) - s(t)), \quad i = k + 1, k + 2, \dots, N,$$

where the feedback gains  $\varepsilon_i$  are positive constants. It can be seen that synchronizing all states  $x_i(t)$  to  $s(t)$  is determined by the dynamics of an isolated node, the coupling strength  $c > 0$ , the inner linking matrix  $\Gamma$ , the feedback gains  $\varepsilon_i \geq 0$ , and the coupling matrix  $A$ .

As discussed in [6, 11, 21], to achieve synchronization of complex dynamical networks, the controllers are generally preferred to be added to the nodes with larger degrees. However, it is also known that, to achieve a certain synchronizability of the network, the feedback gains  $\varepsilon_i$  usually have to be quite large. In [4], when a single controller is used, the coupling strength  $c$  has to be quite large in general. From the view point of realistic applications, these are not expected and sometimes cannot be realized. Practically, a designed control strategy is expected to be effective and also easily implementable. In this paper, for various star-shaped networks and non-regular complex networks, a new concept of cost function is introduced to evaluate the efficiency of the designed controllers. It is found that surprisingly the cost can be much lower by controlling nodes with smaller degrees than controlling nodes with larger degrees. As will be seen, moreover, both the feedback gains  $\varepsilon_i$  and the coupling strength  $c$  can be much smaller than those used in [4, 6, 11, 21].

The outline of this paper is as follows. In Section 2, a new definition of cost function and some mathematical preliminaries are given. Stability of different star-shaped networks controlled by pinning some nodes with small degrees are analyzed in Sections 3 and 4, respectively, where some simulated examples of dynamical networks are compared for illustration and verification. In Section 5, pinning control of non-regular complex dynamical networks of chaotic oscillators is studied through numerical simulations. Finally, Section 6 concludes the paper.

## 2 The cost of pinning control

Denote  $e_i(t) = x_i(t) - s(t)$ , where  $s(t)$  satisfies  $\dot{s}(t) = f(s(t))$ . Then, the error equations of network (1) can be written as

$$\dot{e}_i(t) = f(x_i(t), t) - f(s(t)) + c \sum_{j=1}^N a_{ij} \Gamma e_j(t), \quad i = 1, 2, \dots, N,$$

while the error equations of the controlled network (4) can be written as

$$\dot{e}_i(t) = f(x_i(t), t) - f(s(t)) + c \sum_{j=1}^N \tilde{a}_{ij} \Gamma e_j(t), \quad i = 1, 2, \dots, N, \quad (5)$$

where  $\tilde{a}_{ii} = a_{ii} - \varepsilon_i$ ,  $\varepsilon_i > 0$ ,  $i = k+1, k+2, \dots, N$ , and  $\tilde{a}_{ij} = a_{ij}$  otherwise. Let  $\tilde{A} = (\tilde{a}_{ij}) \in R^{N \times N}$ , and denote  $e(t) = (e_1(t), e_2(t), \dots, e_N(t))^T$ .

Differentiating (5) along  $s(t)$  gives

$$\dot{e}(t) = D(f(s(t)))e(t) + c\Gamma e(t)\tilde{A}^T. \quad (6)$$

By analyzing the matrix  $\tilde{A}^T$ , it is easy to see that all the eigenvalues of  $\tilde{A}^T$  are negative, which are denoted by

$$0 > \lambda_1 \geq \dots \geq \lambda_N.$$

There exists an orthogonal matrix  $U$  such that  $\tilde{A}^T = UJU^{-1}$ , where  $J = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ . Let  $\tilde{e}(t) = e(t)U$ . Then, from (6), one has

$$\dot{\tilde{e}}_i(t) = [Df(s(t)) + c\lambda_i\Gamma]\tilde{e}_i(t), \quad i = 1, \dots, N. \quad (7)$$

Therefore, the local stability problem of network (4) is converted into the stability problem of the  $N$  independent linear systems (7). When the system function of an isolated node and the inner linking matrix  $\Gamma$  are fixed, the stability problem of systems (7) are dependent on the coupling strength  $c$  and the eigenvalues of  $\tilde{A}$ . Clearly, the smaller the eigenvalues of the matrix  $\tilde{A}$  are ( $\lambda_i < 0$ ,  $i = 1, \dots, N$ ), the smaller the coupling strength  $c > 0$  is needed to guarantee the same synchronizability of the network (4), if systems (7) have unbounded synchronization regions [18]. In the following, a cost function is introduced to describe the efficiency of the controllers.

**Definition 1** (Cost Function) Suppose that the feedback gain matrix is  $G = \text{diag}\{\varepsilon_1, \dots, \varepsilon_N\}$ , where  $\varepsilon_i \geq 0$ ,  $i = 1, \dots, N$ , are given as in (4). The Cost Function is defined as

$$CF = c \sum_{i=1}^N \varepsilon_i.$$

**Remark 1** The smaller the  $CF$ , the more efficient a control strategy to achieve the same goal of control, and the easier to be implemented.

In order to discuss the effects of pinning control, the following Lemmas are needed.

**Lemma 2** [8] Let  $A = [a_{ij}] \in C^{m \times n}$  be Hermitian, and let  $a_{nn} \leq \dots \leq a_{22} \leq a_{11}$  be a rearrangement of its diagonal entries in increasing order. Let the eigenvalues of  $A$  be ordered as

$$\lambda_{\min} = \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_2 \leq \lambda_1 = \lambda_{\max}. \quad (8)$$

Then

- (i)  $\lambda_n \leq a_{ii} \leq \lambda_1$  for all  $i = 1, \dots, n$ ,
- (ii)  $a_{11} + a_{22} \leq \lambda_2$ , if  $\lambda_1 = 0$ .

**Remark 2** For diffusive networks, the smaller the  $\lambda_2$ , the easier the synchronization, if the network has an unbound synchronized region [18]. However, Lemma 2 shows that  $\lambda_2$  is related to  $a_{11} + a_{22}$ , where  $a_{11}$  and  $a_{22}$  are determined by two smallest nodes. Therefore, in this case, in order to improve the synchronizability, these small nodes should be pinned.

**Lemma 3** Let  $A \in C^{n \times n}$  be Hermitian, and

$$\tilde{A} = \begin{bmatrix} A_1 & A_{12} \\ A_{21}^T & A_2 + D \end{bmatrix},$$

where  $A_1 \in R^{k \times k}$ ,  $A_{12} \in R^{k \times (N-k)}$ ,  $A_2 \in R^{(N-k) \times (N-k)}$  and  $D = \text{diag}\{\varepsilon_{k+1}, \dots, \varepsilon_N\}$ . Then,  $\tilde{A} < -\alpha I$ ,  $\alpha > 0$ , if and only if  $A_1 < -\alpha I_1$  and  $A_2 + D - A_{12}^T(A_1 + \alpha I_1)^{-1}A_{12} < -\alpha I_2$ , where  $I_1$  and  $I_2$  are identity matrixes with appropriate dimensions.

**Remark 3** Matrix  $D$ , i.e.  $\varepsilon_i$  can be determined by the LMI method [7]. Suppose the eigenvalues of  $A_1$  are  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_k$ . Then  $-\alpha > \beta_1$ ; and if  $\varepsilon_i \rightarrow -\infty$ , then  $-\alpha \rightarrow \beta_1$ .

### 3 Simple star-shaped networks

Now, consider a network consisting of  $N$  identical nodes with a simple star-shaped coupling configuration: there exists a central node which connects to all the other non-central nodes, and there are no direct connections among the non-central nodes. Fig. 1 gives an example of such a network.

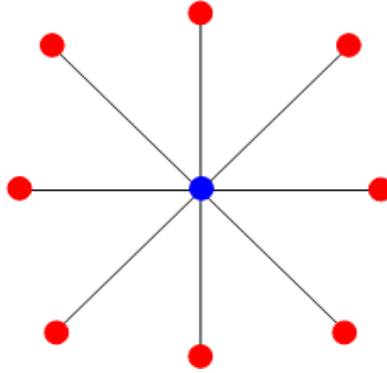


Fig. 1  $N = 9$

The coupling matrix  $A$  of network (1) with a simple star-shaped coupling configuration can be written as

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad (9)$$

where  $A_1 = -N + 1$ ,  $A_{12} = (1, 1, \dots, 1) \in R^{(N-1)}$ ,  $A_{21} = A_{12}^T$ , and  $A_2 = -I_{(N-1) \times (N-1)}$ .

Let the feedback gain matrix be  $G = \text{diag}\{0, \varepsilon, \dots, \varepsilon\} \in R^{N \times N}$ ; that is, the controllers are added to all the non-central nodes for simplicity in this discussion. Then, the coupling matrix of the controlled network (4) is

$$\tilde{A} = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 - \varepsilon I \end{bmatrix}.$$

**Theorem 1** If there are constants  $\varepsilon > 0$  and  $\tilde{k} > 0$ , such that  $N - \tilde{k} - 1 > 0$ ,  $\varepsilon > \tilde{k} - 1$  and  $\varepsilon > \frac{\tilde{k}(N-\tilde{k})}{N-\tilde{k}-1}$ , then  $\lambda_i(\tilde{A}) < -\tilde{k}$ ,  $i = 1, \dots, N$ , where  $\lambda_i(\tilde{A})$ ,  $i = 1, \dots, N$ , are the eigenvalues of the matrix  $\tilde{A}$ .

**Proof:** Consider the matrix

$$\tilde{A} + \tilde{k}I = \begin{bmatrix} A_1 + \tilde{k} & A_{12} \\ A_{21} & (-1 - \varepsilon + \tilde{k})I \end{bmatrix}.$$

By the assumptions of the theorem,  $(-1 - \varepsilon + \tilde{k})I < 0$ . Let  $\bar{A} = (A_1 + \tilde{k}I) - A_{12}((-1 - \varepsilon + \tilde{k})I)^{-1}A_{21}$ . Then

$$\begin{aligned} \bar{A} &= -N + 1 + \tilde{k} + \frac{N-1}{1+\varepsilon-\tilde{k}} \\ &< -N + 1 + \tilde{k} + \frac{N-\tilde{k}-1}{N-1}(N-1) \\ &= 0. \end{aligned}$$

From the Schur complement Lemma,  $\bar{A} + \tilde{k}I < 0$ , which leads to the assertion of the theorem.  $\square$

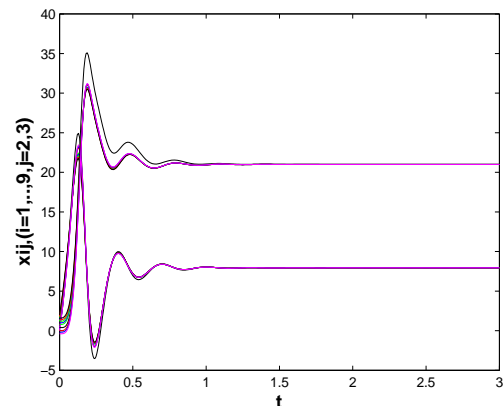
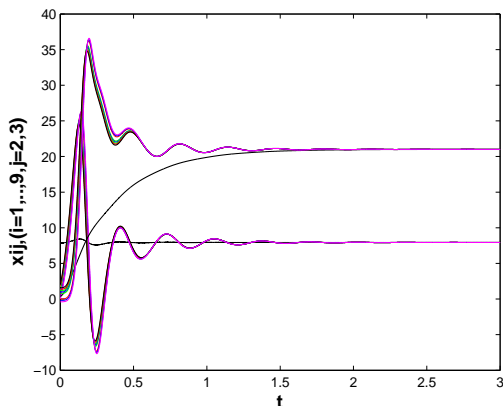
**Remark 4** Theorem 1 shows that under the control strategy of adding controllers to all the non-central nodes, as the constant  $\varepsilon > 0$  increases, the eigenvalues of the matrix  $\tilde{A}$  will approach  $-(N-1)$ . Further, suppose that  $N > 2$  and take  $\tilde{k} = 1$ . Then, based on Theorem 1, to guarantee that  $\lambda_i(\tilde{A}) < -1$ ,  $i = 1, \dots, N$ , the feedback gain  $\varepsilon$  only needs to satisfy

$$\varepsilon > \frac{N-1}{N-2}. \quad (10)$$

In this case, if the coupling strength of network (9) is  $c$ , then the cost function is  $CF = c\varepsilon(N-1)$ . From (10) and through direct calculations, it is easy to see that under the control strategy of adding controllers to the non-central nodes,  $CF > \frac{(N-1)^2}{(N-2)}$  suffices to guarantee that  $\lambda_i(\tilde{A}) < -1$ ,  $i = 1, \dots, N$ .

**Remark 5** Consider the control strategy that a single controller is added to the central node. In this case, the feedback gain matrix is given by  $G = \text{diag}\{\varepsilon, 0, \dots, 0\} \in R^{N \times N}$  and  $CF = c\varepsilon$ . Let  $\lambda_1(\tilde{A})$  represent the largest eigenvalue of matrix  $\tilde{A}$ . Then, based on Lemma 2,  $\lambda_1(\tilde{A}) \geq -1$  no matter how large the  $\varepsilon$ , or equally the cost function  $CF$ , is taken. This shows that in order to guarantee  $\lambda_i$  be small, the small nodes should be pinned.

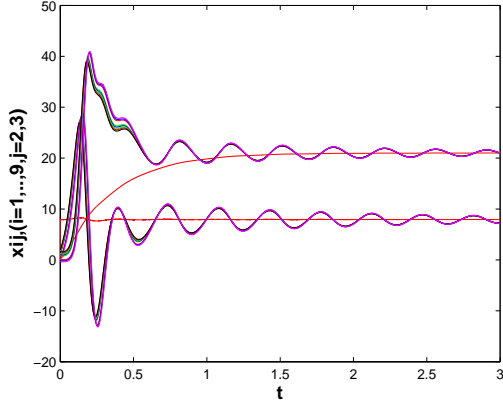
Some simulations on the network (9), shown in Fig. 1, are presented in Figs. 2~3, where the dynamics of an isolated node is a chaotic oscillator [13],  $\Gamma = \text{diag}\{0, 1, 0\}$ , and the synchronized state is set to be  $s(t) = [7.9373 \ 7.9373 \ 21]$ , which is an unstable equilibrium point of an isolated node [14].



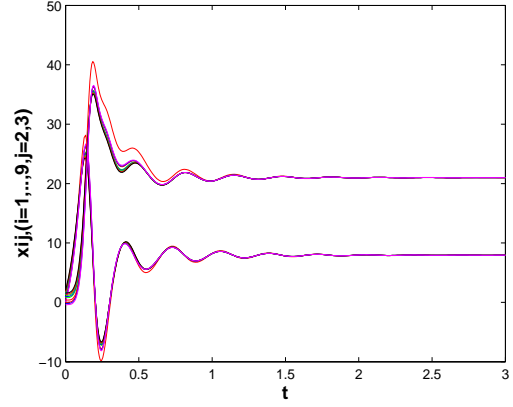
$$(a) CF = c\varepsilon = 300c = 3000.$$

$$(b) CF = c\varepsilon(N - 1) = 1.5(N - 1)c = 120.$$

Fig. 2 Synchronization of the simple star-shaped coupling network (9) with coupling strength  $c = 10$ : (a) adding a single controller to the central node; (b) adding controllers to all the non-central nodes.



$$(a) CF = c\varepsilon = 500c = 3500.$$



$$(b) CF = c\varepsilon(N - 1) = 1.5(N - 1)c = 84.$$

Fig. 3 Synchronization of the simple star-shaped coupling network (9) with coupling strength  $c = 7$ : (a) pinning the central node; (b) pinning all the non-central nodes.

From Figs. 2~3, it can be seen that even with both a small coupling strength  $c$  and a small cost function  $CF$ , the synchronization of network (9) can be faster to achieve when the controllers are added to all the non-central nodes (nodes with smaller degrees) than the case that a single controller is added to the central node (the node with a larger degree). Although the more controllers are needed for pinning nodes with smaller degrees, the total cost is still lower and the control effect is better in comparison.

## 4 Clusters of star-shaped networks with global coupling

Consider a network consisting of  $N$  identical nodes in clusters of star-shaped coupling configuration: there are  $k$  central nodes which are connected to each other; each central node may have different numbers of non-central nodes attached to it; there are no direct connections among the non-central nodes; and any central node has no direct connection to a non-central node attached to another central node. Fig. 4 gives an example of such a network for the case of  $k = 3$ .

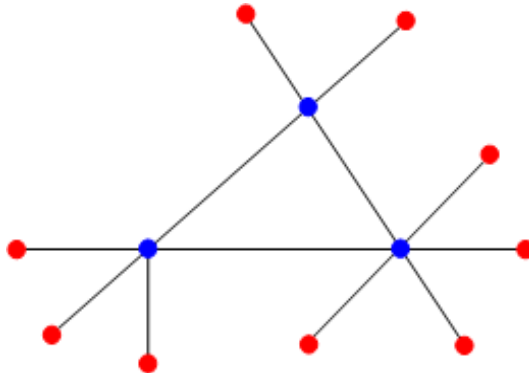


Fig. 4 A cluster of star-shaped network with  $k = 3$

Without loss of generality, let the first  $k$  nodes be central nodes, each connecting to  $n_i$  non-central nodes, where  $n_i$  satisfy that  $n_1 \leq n_2 \leq \dots \leq n_k$ . Clearly,  $N = k + \sum_{i=1}^k n_i$ . The coupling matrix  $A$  of network (1) in such clusters of star-shaped coupling configuration can be written as

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad (11)$$

where  $A_1 \in R^{k \times k}$ ,  $A_{12} \in R^{k \times (N-k)}$ ,  $A_{21} = A_{12}^T \in R^{(N-k) \times k}$ , and  $A_2 \in R^{(N-k) \times (N-k)}$ . Let  $\alpha_i = (1, 1, \dots, 1) \in R^{n_i}$ ,  $i = 1, \dots, k$ . Then

$$A_1 = \begin{bmatrix} -k+1-n_1 & 1 & \dots & 1 \\ 1 & -k+1-n_2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -k+1-n_k \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_k \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} \alpha_1^T & & & \\ & \alpha_2^T & & \\ & & \ddots & \\ & & & \alpha_k^T \end{bmatrix}, \quad A_2 = -I_{(N-k)}.$$

Let the feedback gain matrix be  $G = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon I_{(N-k)} \end{bmatrix}$ . Then, the corresponding coupling matrix  $\tilde{A}$  of the controlled network (4) is

$$\tilde{A} = A - G. \quad (12)$$

**Theorem 2** If there are constants  $\varepsilon > 0$  and  $\tilde{k} > 0$ , such that  $n_1 > \tilde{k}$ ,  $\varepsilon > \tilde{k} - 1$  and  $\varepsilon > \frac{\tilde{k}(n_1+1-\tilde{k})}{n_1-\tilde{k}}$ , then  $\lambda_i(\tilde{A}) < -\tilde{k}$ ,  $i = 1, \dots, N$ .

**Proof:** Consider the matrix

$$\tilde{A} + \tilde{k}I = \begin{bmatrix} A_1 + \tilde{k}I & A_{12} \\ A_{21} & (-1 - \varepsilon + \tilde{k})I \end{bmatrix}.$$

By the assumptions of the theorem,  $(-1 - \varepsilon + \tilde{k})I < 0$ . Let  $\bar{A} = (A_1 + \tilde{k}I) - A_{12}((-1 - \varepsilon + \tilde{k})I)^{-1}A_{21}$ . Then  $\bar{A}$  can be expressed as

$$\bar{A} = \begin{bmatrix} -k+1-n_1 + \tilde{k} + \frac{n_1}{1+\varepsilon-\tilde{k}} & 1 & \dots & 1 \\ 1 & -k+1-n_2 + \tilde{k} + \frac{n_2}{1+\varepsilon-\tilde{k}} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -k+1-n_k + \tilde{k} + \frac{n_k}{1+\varepsilon-\tilde{k}} \end{bmatrix}.$$

Let  $m_i = -k+1-n_i + \tilde{k} + \frac{n_i}{1+\varepsilon-\tilde{k}} + (k-1) \cdot 1$ ,  $i = 1, 2, \dots, k$ . Then

$$\begin{aligned} m_i &= -n_i + \tilde{k} + \frac{n_i}{1+\varepsilon-\tilde{k}} \\ &< -n_i + \tilde{k} + \frac{n_1-\tilde{k}}{n_1}n_i \\ &= n_i\left(\frac{-\tilde{k}}{n_1}\right) + \tilde{k} \\ &= \tilde{k}\left(1 - \frac{n_i}{n_1}\right) \leq 0. \end{aligned}$$

According to the Gerschgorin disc theorem [8],  $\bar{A} < 0$ . By the Schur complement Lemma,  $\tilde{A} + \tilde{k}I < 0$ , which leads to the assertion of the theorem.  $\square$

Some simulations on the network (11) shown in Fig. 4 are presented in Fig. 5, where the dynamics of an isolated node, the inner liking matrix  $\Gamma$ , and the synchronized state are the same as that given in Section 2.

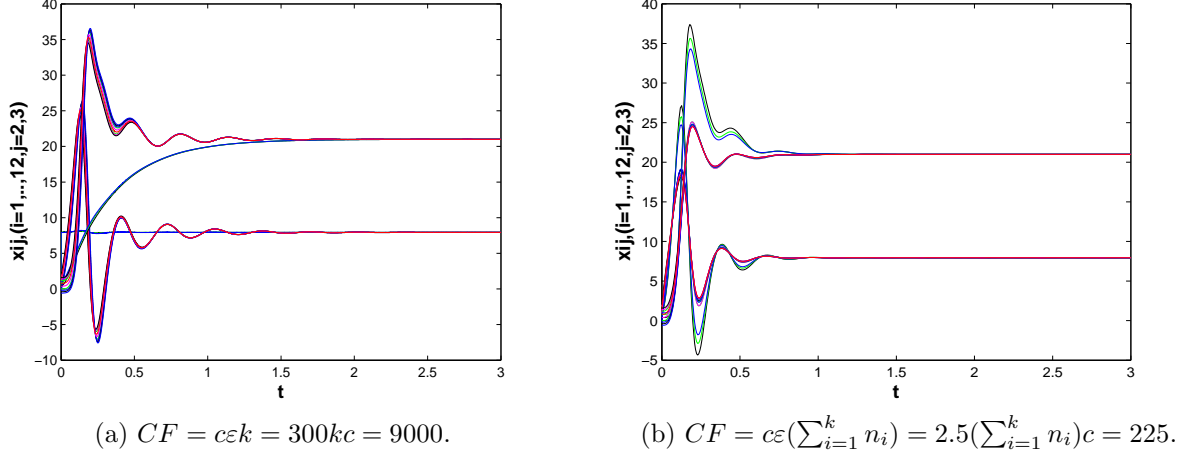


Fig. 5 Synchronization of network (11) with coupling strength  $c = 10$ : (a) pinning  $k$  central nodes; (b) pinning all the non-central nodes.

From Fig. 5, it can be seen that although all the non-central nodes are pinned, the cost is lower and the effect is better than the case where the central nodes are pinned.

By using the Kronecker product, the controlled network (5) can be rewritten as

$$\dot{e}(t) = F(x, s) + c(A - G) \otimes \Gamma e(t), \quad (13)$$

where  $e(t) = (e_1^T(t), \dots, e_N^T(t))^T$ ,  $F(x, s) = ((f(x_1(t)) - f(s(t)))^T, \dots, (f(x_N(t)) - f(s(t)))^T)^T$ ,  $G = \text{diag}\{\underbrace{0, \dots, 0}_k, \underbrace{\varepsilon, \dots, \varepsilon}_{N-k}\} \in R^{N \times N}$  and  $\Gamma = \text{diag}\{r_1, r_2, \dots, r_n\}$ ,  $r_i = 1$  or  $0$ ,  $i = 1, \dots, n$ .

**Theorem 3** If there are a positive diagonal matrix  $P$  and constants  $\mu > 0$ ,  $\varepsilon > 0$ ,  $\tilde{k} > 0$ ,  $c > 0$ , such that  $n_1 > \tilde{k}$ ,  $\varepsilon > \tilde{k} - 1$ ,  $\varepsilon > \frac{\tilde{k}(n_1+1-\tilde{k})}{n_1-\tilde{k}}$  and

$$(x - y)^T P(f(x, t) - f(y, t) - c\tilde{k}\Gamma(x - y)) \leq -\mu(x - y)^T(x - y),$$

then the controlled network (4) is globally exponentially synchronized to  $s(t)$ .

**Proof:** Choose a Lyapunov function as  $V(t) = \frac{1}{2}e^T(t)(I_N \otimes P)e(t)$ . Its time derivative is

$$\begin{aligned} \dot{V}(t) &= e^T(t)(I_N \otimes P)\dot{e}(t) \\ &= e^T(t)(I_N \otimes P)(F(x, s) + c(A - G) \otimes \Gamma e(t)) \\ &= e^T(t)(I_N \otimes P)F(x, s) + ce^T(t)((A - G) \otimes P\Gamma)e(t) \\ &= e^T(t)(I_N \otimes P)(F(x, s) - c(\tilde{k}I_n \otimes \Gamma)e^T(t)) + ce^T(t)((A - G) + \tilde{k}I_n) \otimes P\Gamma e(t). \end{aligned}$$

By Theorem 2,  $(A - G) + \tilde{k}I_n < 0$ . So,

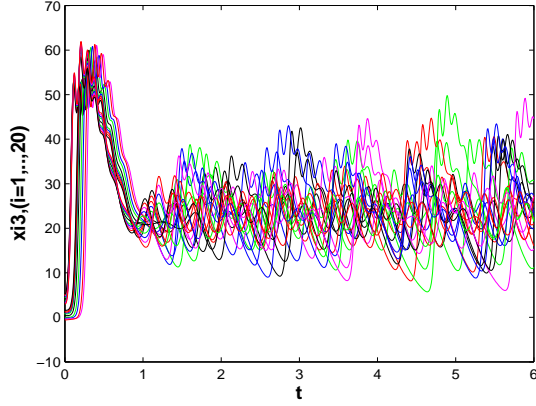
$$\begin{aligned} \dot{V}(t) &\leq -\mu \sum_{i=1}^N e_i^T(t)e_i(t) \leq -\mu \sum_{i=1}^N \frac{1}{\max\{P_j\}_{1 \leq j \leq n}} e_i^T(t)P e_i(t) \\ &= -\frac{\mu}{\max\{P_j\}_{1 \leq j \leq n}} e_i^T(t)(I_n \otimes P)e_i(t) = -\frac{\mu}{\max\{P_j\}_{1 \leq j \leq n}} V(t), \end{aligned}$$

and the theorem is thus proved.  $\square$

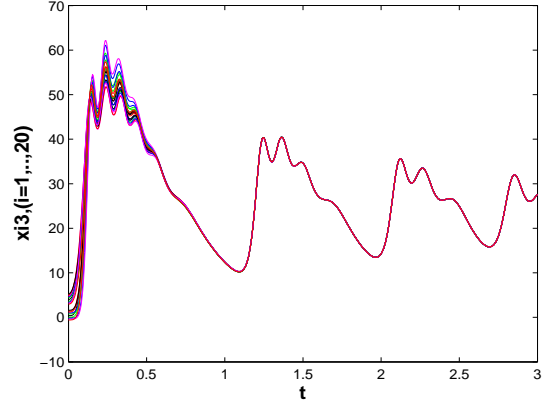


## 5 Some non-regular complex networks

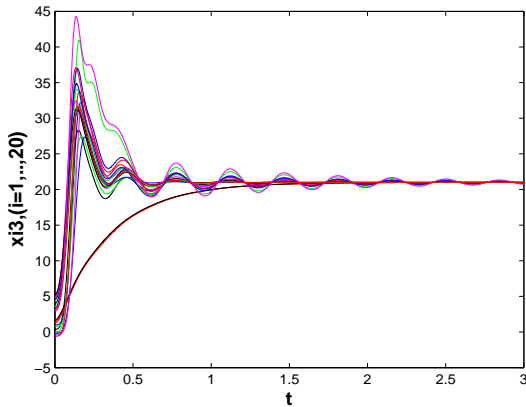
In this section, first, a non-regular coupled network consisting of 20 nodes is generated following the procedure of the well-known BA model. Fig. 6 shows the synchronization of the network with controllers being added to the three “biggest” nodes of degrees 15, 13 and 10, respectively. The dynamics of an isolated node, the inner liking matrix  $\Gamma$ , and the synchronized state are the same as that given in Section 2. From Figs. 6(c) and 6(d), it can be seen that the network does not synchronize any faster, although a larger feedback gain is used. In Fig. 7, a different control scheme is applied: eleven nodes with smaller degrees in the network are pinned with coupling strength  $c = 6$  and feedback gain  $\varepsilon = 5$ , yielding cost function  $CF = 330$ . Compared with Fig. 6, a better control performance is obtained with a much smaller cost function. Figs. 8 and 9 also show that it is more efficient by pinning nodes with smaller degrees than pinning nodes with larger degrees. As shown in Fig. 8, to achieve a similar synchronization effect, a much smaller feedback gain is needed in pinning “smaller” nodes than pinning “larger” ones, thereby the cost function is also smaller in the former control scheme than that in the latter. Fig. 9 shows that although the coupling strength, the feedback gain and the cost function are similar, the synchronization effect is better in pinning “smaller” nodes than pinning “larger” ones.



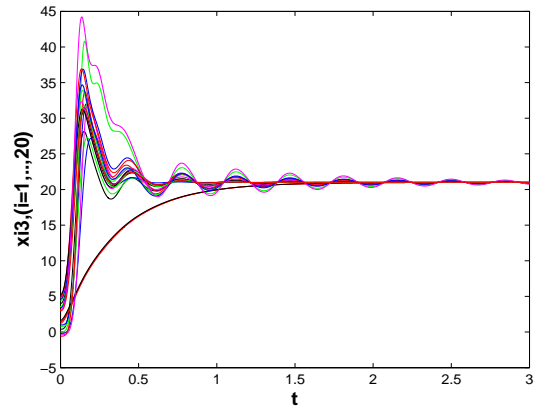
(a)



(b)



(c)



(d)

Fig. 6 Pinning the three “biggest” nodes with degrees 15, 13 and 10 in a 20-node non-regular coupled network: (a)  $c = 0, \varepsilon = 0, CF = 0$ . (b)  $c = 6, \varepsilon = 0, CF = 0$ . (c)  $c = 6, \varepsilon = 500, CF = 9000$ . (d)  $c = 6, \varepsilon = 1000, CF = 18000$ .

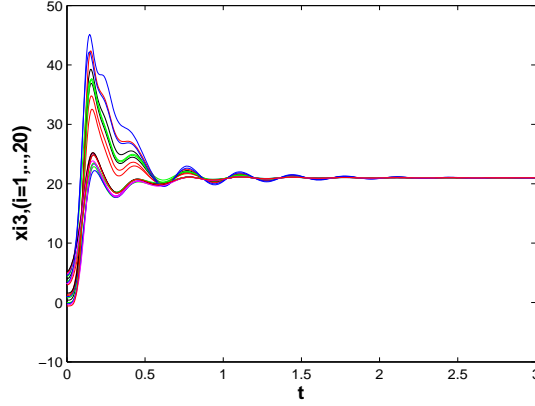
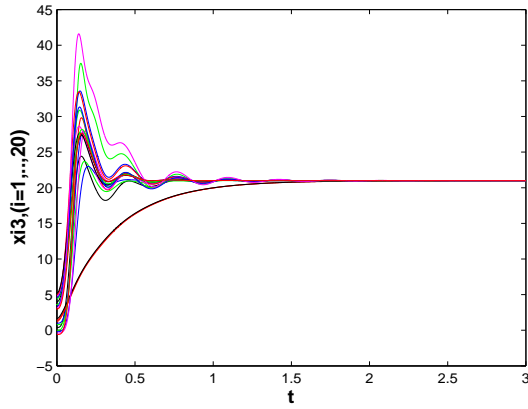
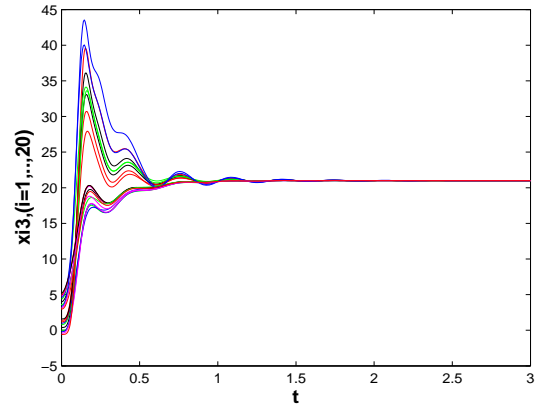


Fig. 7 Pinning the eleven “smaller” nodes in a 20-node non-regular coupled network:  $c = 6, \varepsilon = 5, CF = 330$ .

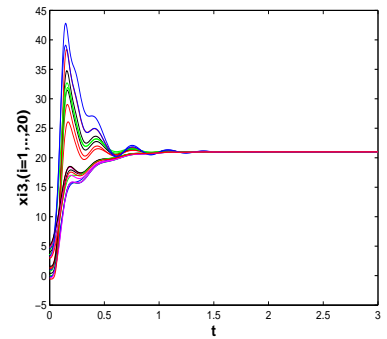
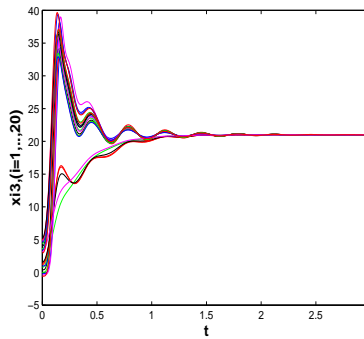
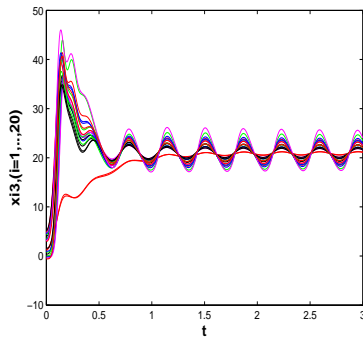


(a)  $c = 8, \varepsilon = 500, CF = 12000$ .



(b)  $c = 6, \varepsilon = 8, CF = 528$ .

Fig. 8 (a) Pinning the three “biggest” nodes with degrees 15, 13 and 10. (b) Pinning the eleven “smaller” nodes.



(a)  $c = 6$ ,  $\varepsilon = 55$ ,  $CF = 660$ .      (b)  $c = 6$ ,  $\varepsilon = 22$ ,  $CF = 660$ .      (c)  $c = 6$ ,  $\varepsilon = 10$ ,  $CF = 660$ .

Fig. 9 (a) Pinning the two “biggest” nodes with degrees 15 and 13. (b) Pinning the three “biggest” nodes with degrees 15, 13 and 10, and the two “smallest” nodes with the same degrees 3. (c) Pinning the eleven “smaller” nodes.

It seems more efficient to use the control scheme of pinning nodes with smaller degrees than pinning nodes with larger degrees for achieving the desired synchronous states of some non-regular coupled dynamical networks. This is an interesting phenomenon that was not noticed before.

## 6 Conclusions

In this paper, pinning control to achieve synchronization of complex dynamical networks is further investigated. In contrary to the general perception, the nodes with smaller degrees play an important role in the synchronizability of some networks. It has been shown by computer simulations in this paper that to achieve a similar synchronization effect on the networks considered, a smaller coupling strength, a smaller feedback gain and thus a lower cost function are needed in the control scheme of pinning nodes with smaller degrees comparing with those needed in pinning nodes with larger degrees. In other words, it seems more efficient to use the control scheme of pinning nodes with smaller degrees than that of pinning nodes with larger degrees for some dynamical networks to achieve synchronization, an interesting phenomenon that deserves further investigation in the future.

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# Cost and Effects of Pinning Control for Network Synchronization \*

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**Abstract.** In this paper, the problem of pinning control for synchronization of complex dynamical networks is discussed. A cost function of the controlled network is defined by the feedback gain and the coupling strength of the network. An interesting result is that lower cost is achieved by the control scheme of pinning nodes with smaller degrees. Some rigorous mathematical analysis is presented for achieving lower cost in the synchronization of different star-shaped networks. Numerical simulations on some non-regular complex networks generated by the Barabási-Albert model and various star-shaped networks are shown for verification and illustration.

**Keywords:** complex dynamical network, pinning control, exponential stability.

## 1 Introduction and problem formulation

Complex networks are currently being studied across many fields of sciences, including physics, chemistry, biology, mathematics, sociology and engineering [1, 2, 3, 5, 9, 15, 17, 19]. A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. Examples of complex networks include the Internet, food webs, cellular neural networks, biological neural networks, electrical power grids, telephone cell graphs, etc. Recently, synchronization of complex networks of dynamical systems has received a great deal of attention from the nonlinear dynamics community [10, 12, 16, 18, 20]. A special control strategy called *pinning control* is used to achieve synchronization of complex networks; that is, only a fraction of the nodes or even a single node is controlled over the whole network [4, 6, 11, 21]. This control method has become a common technique for control, stabilization and synchronization of coupled dynamical systems. In general, different nodes have different degrees in a network, thus a natural question is how different the effect would be when nodes with different degrees are pinned.

Consider a dynamical network consisting of  $N$  identical and diffusively coupled nodes, with each node being an  $n$ -dimensional dynamical system. The state equations of the network are

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t), \quad i = 1, 2, \dots, N, \quad (1)$$

where  $f(\cdot)$  is the dynamical function of an isolated node,  $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in R^n$  are the state variables of node  $i$ , constant  $c > 0$  represents the coupling strength, and  $\Gamma \in R^{n \times n}$  is the

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inner linking matrix. Moreover, the coupling matrix  $A = (a_{ij}) \in R^{N \times N}$  represents the coupling configuration of the network: If there is a connection between node  $i$  and node  $j$  ( $i \neq j$ ), then  $a_{ij} = a_{ji} = 1$ ; otherwise,  $a_{ij} = a_{ji} = 0$  ( $i \neq j$ ); the diagonal entries of  $A$  are defined by

$$a_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij}, \quad i = 1, 2, \dots, N. \quad (2)$$

Suppose that the network is connected in the sense of having no isolated clusters. Then, the coupling matrix  $A$  is irreducible. From Lemma 2 of [20], it can be proved that zero is an eigenvalue of  $A$  with multiplicity one and all the other eigenvalues of  $A$  are strictly negative.

Network (1) is said to achieve (asymptotical) synchronization if

$$x_1(t) \rightarrow x_2(t) \rightarrow \dots \rightarrow x_N(t) \rightarrow s(t), \quad \text{as } t \rightarrow \infty, \quad (3)$$

where, because of the diffusive coupling configuration,  $s(t)$  is a solution of an isolated node, which can be an equilibrium, a periodic or a chaotic orbit. As shown in [4, 6, 11, 21], this can be achieved by controlling several nodes (or even only one node) of the network. Without loss of generality, suppose that the controllers are added on the last  $N - k$  nodes of the network, so that the equations of the controlled network can be written as

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t), \quad i = 1, 2, \dots, k, \quad (4)$$

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j=1}^N a_{ij} \Gamma x_j(t) - c \varepsilon_i \Gamma (x_i(t) - s(t)), \quad i = k + 1, k + 2, \dots, N,$$

where the feedback gains  $\varepsilon_i$  are positive constants. It can be seen that synchronizing all states  $x_i(t)$  to  $s(t)$  is determined by the dynamics of an isolated node, the coupling strength  $c > 0$ , the inner linking matrix  $\Gamma$ , the feedback gains  $\varepsilon_i \geq 0$ , and the coupling matrix  $A$ .

As discussed in [6, 11, 21], to achieve synchronization of complex dynamical networks, the controllers are generally preferred to be added to the nodes with larger degrees. However, it is also known that, to achieve a certain synchronizability of the network, the feedback gains  $\varepsilon_i$  usually have to be quite large. In [4], when a single controller is used, the coupling strength  $c$  has to be quite large in general. From the view point of realistic applications, these are not expected and sometimes cannot be realized. Practically, a designed control strategy is expected to be effective and also easily implementable. In this paper, for various star-shaped networks and non-regular complex networks, a new concept of cost function is introduced to evaluate the efficiency of the designed controllers. It is found that surprisingly the cost can be much lower by controlling nodes with smaller degrees than controlling nodes with larger degrees. As will be seen, moreover, both the feedback gains  $\varepsilon_i$  and the coupling strength  $c$  can be much smaller than those used in [4, 6, 11, 21].

The outline of this paper is as follows. In Section 2, a new definition of cost function and some mathematical preliminaries are given. Stability of different star-shaped networks controlled by pinning some nodes with small degrees are analyzed in Sections 3, 4 and 5, respectively, where some simulated examples of dynamical networks are compared for illustration and verification. In Section 6, pinning control of non-regular complex dynamical networks of chaotic oscillators is studied through numerical simulations. Finally, Section 7 concludes the paper.

## 2 The cost of pinning control

Denote  $e_i(t) = x_i(t) - s(t)$ , where  $s(t)$  satisfies  $\dot{s}(t) = f(s(t))$ . Then, the error equations of network (1) can be written as

$$\dot{e}_i(t) = f(x_i(t), t) - f(s(t)) + c \sum_{j=1}^N a_{ij} \Gamma e_j(t), \quad i = 1, 2, \dots, N,$$

while the error equations of the controlled network (4) can be written as

$$\dot{e}_i(t) = f(x_i(t), t) - f(s(t)) + c \sum_{j=1}^N \tilde{a}_{ij} \Gamma e_j(t), \quad i = 1, 2, \dots, N, \quad (5)$$

where  $\tilde{a}_{ii} = a_{ii} - \varepsilon_i$ ,  $\varepsilon_i > 0$ ,  $i = k+1, k+2, \dots, N$ , and  $\tilde{a}_{ij} = a_{ij}$  otherwise. Let  $\tilde{A} = (\tilde{a}_{ij}) \in R^{N \times N}$ , and denote  $e(t) = (e_1(t), e_2(t), \dots, e_N(t))^T$ .

Differentiating (5) along  $s(t)$  gives

$$\dot{e}(t) = D(f(s(t)))e(t) + c\Gamma e(t)\tilde{A}^T. \quad (6)$$

By analyzing the matrix  $\tilde{A}^T$ , it is easy to see that all the eigenvalues of  $\tilde{A}^T$  are negative, which are denoted by

$$0 > \lambda_1 \geq \dots \geq \lambda_N.$$

There exists an orthogonal matrix  $U$  such that  $\tilde{A}^T = UJU^{-1}$ , where  $J = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ . Let  $\tilde{e}(t) = e(t)U$ . Then, from (6), one has

$$\dot{\tilde{e}}_i(t) = [Df(s(t)) + c\lambda_i\Gamma]\tilde{e}_i(t), \quad i = 1, \dots, N. \quad (7)$$

Therefore, the local stability problem of network (4) is converted into the stability problem of the  $N$  independent linear systems (7). When the system function of an isolated node and the inner linking matrix  $\Gamma$  are fixed, the stability problem of systems (7) are dependent on the coupling strength  $c$  and the eigenvalues of  $\tilde{A}$ . Clearly, the smaller the eigenvalues of the matrix  $\tilde{A}$  are ( $\lambda_i < 0$ ,  $i = 1, \dots, N$ ), the smaller the coupling strength  $c > 0$  is needed to guarantee the same synchronizability of the network (4), if systems (7) have unbounded synchronization regions [18]. In the following, a cost function is introduced to describe the efficiency of the controllers.

**Definition 1** (Cost Function) Suppose that the feedback gain matrix is  $G = \text{diag}\{\varepsilon_1, \dots, \varepsilon_N\}$ , where  $\varepsilon_i \geq 0$ ,  $i = 1, \dots, N$ , are given as in (4). The Cost Function is defined as

$$CF = c \sum_{i=1}^N \varepsilon_i.$$

**Remark 1** The smaller the  $CF$ , the more efficient a control strategy to achieve the same goal of control, and the easier to be implemented.

In order to discuss the effects of pinning control, the following Lemmas are needed.

**Lemma 2** [8] Let  $A = [a_{ij}] \in C^{m \times n}$  be Hermitian, and let  $a_{nn} \leq \dots \leq a_{22} \leq a_{11}$  be a rearrangement of its diagonal entries in increasing order. Let the eigenvalues of  $A$  be ordered as

$$\lambda_{\min} = \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_2 \leq \lambda_1 = \lambda_{\max}. \quad (8)$$

Then

- (i)  $\lambda_n \leq a_{ii} \leq \lambda_1$  for all  $i = 1, \dots, n$ ,
- (ii)  $a_{11} + a_{22} \leq \lambda_2$ , if  $\lambda_1 = 0$ .

**Remark 2** For diffusive networks, the smaller the  $\lambda_2$ , the easier the synchronization, if the network has an unbound synchronized region [18]. However, Lemma 2 shows that  $\lambda_2$  is related to  $a_{11} + a_{22}$ , where  $a_{11}$  and  $a_{22}$  are determined by two smallest nodes. Therefore, in this case, in order to improve the synchronizability, these small nodes should be pinned.

**Lemma 3** Let  $A \in C^{n \times n}$  be Hermitian, and

$$\tilde{A} = \begin{bmatrix} A_1 & A_{12} \\ A_{21}^T & A_2 + D \end{bmatrix},$$

where  $A_1 \in R^{k \times k}$ ,  $A_{12} \in R^{k \times (N-k)}$ ,  $A_2 \in R^{(N-k) \times (N-k)}$  and  $D = \text{diag}\{\varepsilon_{k+1}, \dots, \varepsilon_N\}$ . Then,  $\tilde{A} < -\alpha I$ ,  $\alpha > 0$ , if and only if  $A_1 < -\alpha I_1$  and  $A_2 + D - A_{12}^T(A_1 + \alpha I_1)^{-1}A_{12} < -\alpha I_2$ , where  $I_1$  and  $I_2$  are identity matrixes with appropriate dimensions.

**Remark 3** Matrix  $D$ , i.e.  $\varepsilon_i$  can be determined by the LMI method [7]. Suppose the eigenvalues of  $A_1$  are  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_k$ . Then  $-\alpha > \beta_1$ ; and if  $\varepsilon_i \rightarrow -\infty$ , then  $-\alpha \rightarrow \beta_1$ .

### 3 Simple star-shaped networks

Now, consider a network consisting of  $N$  identical nodes with a simple star-shaped coupling configuration: there exists a central node which connects to all the other non-central nodes, and there are no direct connections among the non-central nodes. Fig. 1 gives an example of such a network.

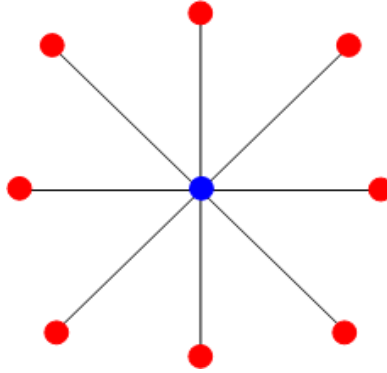


Fig. 1  $N = 9$

The coupling matrix  $A$  of network (1) with a simple star-shaped coupling configuration can be written as

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad (9)$$

where  $A_1 = -N + 1$ ,  $A_{12} = (1, 1, \dots, 1) \in R^{(N-1)}$ ,  $A_{21} = A_{12}^T$ , and  $A_2 = -I_{(N-1) \times (N-1)}$ .

Let the feedback gain matrix be  $G = \text{diag}\{0, \varepsilon, \dots, \varepsilon\} \in R^{N \times N}$ ; that is, the controllers are added to all the non-central nodes for simplicity in this discussion. Then, the coupling matrix of the controlled network (4) is

$$\tilde{A} = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 - \varepsilon I \end{bmatrix}.$$



**Theorem 1** If there are constants  $\varepsilon > 0$  and  $\tilde{k} > 0$ , such that  $N - \tilde{k} - 1 > 0$ ,  $\varepsilon > \tilde{k} - 1$  and  $\varepsilon > \frac{\tilde{k}(N-\tilde{k})}{N-\tilde{k}-1}$ , then  $\lambda_i(\tilde{A}) < -\tilde{k}$ ,  $i = 1, \dots, N$ , where  $\lambda_i(\tilde{A})$ ,  $i = 1, \dots, N$ , are the eigenvalues of the matrix  $\tilde{A}$ .

**Proof:** Consider the matrix

$$\tilde{A} + \tilde{k}I = \begin{bmatrix} A_1 + \tilde{k} & A_{12} \\ A_{21} & (-1 - \varepsilon + \tilde{k})I \end{bmatrix}.$$

By the assumptions of the theorem,  $(-1 - \varepsilon + \tilde{k})I < 0$ . Let  $\bar{A} = (A_1 + \tilde{k}I) - A_{12}((-1 - \varepsilon + \tilde{k})I)^{-1}A_{21}$ . Then

$$\begin{aligned} \bar{A} &= -N + 1 + \tilde{k} + \frac{N-1}{1+\varepsilon-\tilde{k}} \\ &< -N + 1 + \tilde{k} + \frac{N-\tilde{k}-1}{N-1}(N-1) \\ &= 0. \end{aligned}$$

From the Schur complement Lemma,  $\bar{A} + \tilde{k}I < 0$ , which leads to the assertion of the theorem.  $\square$

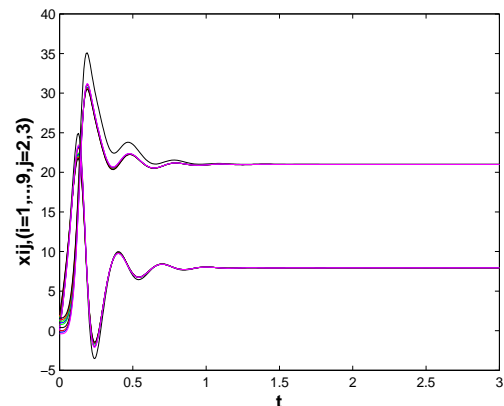
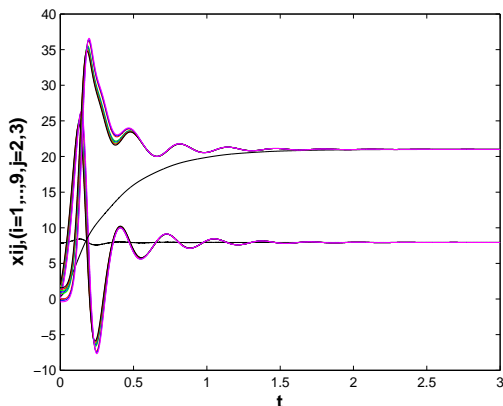
**Remark 4** Theorem 1 shows that under the control strategy of adding controllers to all the non-central nodes, as the constant  $\varepsilon > 0$  increases, the eigenvalues of the matrix  $\tilde{A}$  will approach  $-(N-1)$ . Further, suppose that  $N > 2$  and take  $\tilde{k} = 1$ . Then, based on Theorem 1, to guarantee that  $\lambda_i(\tilde{A}) < -1$ ,  $i = 1, \dots, N$ , the feedback gain  $\varepsilon$  only needs to satisfy

$$\varepsilon > \frac{N-1}{N-2}. \quad (10)$$

In this case, if the coupling strength of network (9) is  $c$ , then the cost function is  $CF = c\varepsilon(N-1)$ . From (10) and through direct calculations, it is easy to see that under the control strategy of adding controllers to the non-central nodes,  $CF > \frac{(N-1)^2}{(N-2)}$  suffices to guarantee that  $\lambda_i(\tilde{A}) < -1$ ,  $i = 1, \dots, N$ .

**Remark 5** Consider the control strategy that a single controller is added to the central node. In this case, the feedback gain matrix is given by  $G = \text{diag}\{\varepsilon, 0, \dots, 0\} \in R^{N \times N}$  and  $CF = c\varepsilon$ . Let  $\lambda_1(\tilde{A})$  represent the largest eigenvalue of matrix  $\tilde{A}$ . Then, based on Lemma 2,  $\lambda_1(\tilde{A}) \geq -1$  no matter how large the  $\varepsilon$ , or equally the cost function  $CF$ , is taken. This shows that in order to guarantee  $\lambda_i$  be small, the small nodes should be pinned.

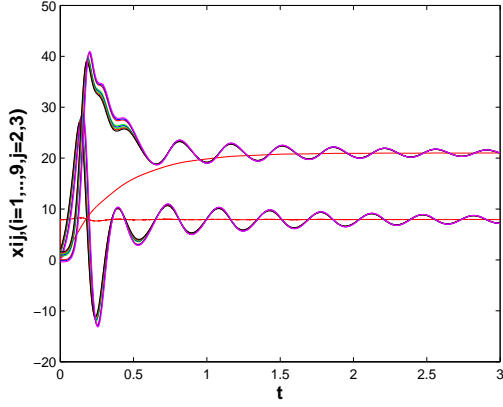
Some simulations on the network (9), shown in Fig. 1, are presented in Figs. 2~3, where the dynamics of an isolated node is a chaotic oscillator [13],  $\Gamma = \text{diag}\{0, 1, 0\}$ , and the synchronized state is set to be  $s(t) = [7.9373 \ 7.9373 \ 21]$ , which is an unstable equilibrium point of an isolated node [14].



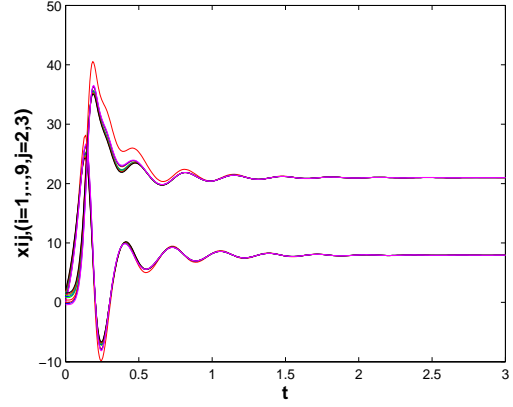
$$(a) CF = c\varepsilon = 300c = 3000.$$

$$(b) CF = c\varepsilon(N - 1) = 1.5(N - 1)c = 120.$$

Fig. 2 Synchronization of the simple star-shaped coupling network (9) with coupling strength  $c = 10$ : (a) adding a single controller to the central node; (b) adding controllers to all the non-central nodes.



$$(a) CF = c\varepsilon = 500c = 3500.$$



$$(b) CF = c\varepsilon(N - 1) = 1.5(N - 1)c = 84.$$

Fig. 3 Synchronization of the simple star-shaped coupling network (9) with coupling strength  $c = 7$ : (a) pinning the central node; (b) pinning all the non-central nodes.

From Figs. 2~3, it can be seen that even with both a small coupling strength  $c$  and a small cost function  $CF$ , the synchronization of network (9) can be faster to achieve when the controllers are added to all the non-central nodes (nodes with smaller degrees) than the case that a single controller is added to the central node (the node with a larger degree). Although the more controllers are needed for pinning nodes with smaller degrees, the total cost is still lower and the control effect is better in comparison.

## 4 Clusters of star-shaped networks with global coupling

Consider a network consisting of  $N$  identical nodes in clusters of star-shaped coupling configuration: there are  $k$  central nodes which are connected to each other; each central node may have different numbers of non-central nodes attached to it; there are no direct connections among the non-central nodes; and any central node has no direct connection to a non-central node attached to another central node. Fig. 4 gives an example of such a network for the case of  $k = 3$ .

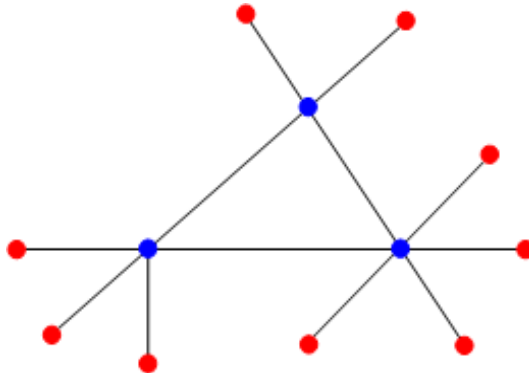


Fig. 4 A cluster of star-shaped network with  $k = 3$

Without loss of generality, let the first  $k$  nodes be central nodes, each connecting to  $n_i$  non-central nodes, where  $n_i$  satisfy that  $n_1 \leq n_2 \leq \dots \leq n_k$ . Clearly,  $N = k + \sum_{i=1}^k n_i$ . The coupling matrix  $A$  of network (1) in such clusters of star-shaped coupling configuration can be written as

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad (11)$$

where  $A_1 \in R^{k \times k}$ ,  $A_{12} \in R^{k \times (N-k)}$ ,  $A_{21} = A_{12}^T \in R^{(N-k) \times k}$ , and  $A_2 \in R^{(N-k) \times (N-k)}$ . Let  $\alpha_i = (1, 1, \dots, 1) \in R^{n_i}$ ,  $i = 1, \dots, k$ . Then

$$A_1 = \begin{bmatrix} -k+1-n_1 & 1 & \dots & 1 \\ 1 & -k+1-n_2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -k+1-n_k \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_k \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} \alpha_1^T & & & \\ & \alpha_2^T & & \\ & & \ddots & \\ & & & \alpha_k^T \end{bmatrix}, \quad A_2 = -I_{(N-k)}.$$

Let the feedback gain matrix be  $G = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon I_{(N-k)} \end{bmatrix}$ . Then, the corresponding coupling matrix  $\tilde{A}$  of the controlled network (4) is

$$\tilde{A} = A - G. \quad (12)$$

**Theorem 2** If there are constants  $\varepsilon > 0$  and  $\tilde{k} > 0$ , such that  $n_1 > \tilde{k}$ ,  $\varepsilon > \tilde{k} - 1$  and  $\varepsilon > \frac{\tilde{k}(n_1+1-\tilde{k})}{n_1-\tilde{k}}$ , then  $\lambda_i(\tilde{A}) < -\tilde{k}$ ,  $i = 1, \dots, N$ .

**Proof:** Consider the matrix

$$\tilde{A} + \tilde{k}I = \begin{bmatrix} A_1 + \tilde{k}I & A_{12} \\ A_{21} & (-1 - \varepsilon + \tilde{k})I \end{bmatrix}.$$

By the assumptions of the theorem,  $(-1 - \varepsilon + \tilde{k})I < 0$ . Let  $\bar{A} = (A_1 + \tilde{k}I) - A_{12}((-1 - \varepsilon + \tilde{k})I)^{-1}A_{21}$ . Then  $\bar{A}$  can be expressed as

$$\bar{A} = \begin{bmatrix} -k+1-n_1 + \tilde{k} + \frac{n_1}{1+\varepsilon-\tilde{k}} & 1 & \dots & 1 \\ 1 & -k+1-n_2 + \tilde{k} + \frac{n_2}{1+\varepsilon-\tilde{k}} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -k+1-n_k + \tilde{k} + \frac{n_k}{1+\varepsilon-\tilde{k}} \end{bmatrix}.$$

Let  $m_i = -k+1-n_i + \tilde{k} + \frac{n_i}{1+\varepsilon-\tilde{k}} + (k-1) \cdot 1$ ,  $i = 1, 2, \dots, k$ . Then

$$\begin{aligned} m_i &= -n_i + \tilde{k} + \frac{n_i}{1+\varepsilon-\tilde{k}} \\ &< -n_i + \tilde{k} + \frac{n_1-\tilde{k}}{n_1}n_i \\ &= n_i\left(\frac{-\tilde{k}}{n_1}\right) + \tilde{k} \\ &= \tilde{k}\left(1 - \frac{n_i}{n_1}\right) \leq 0. \end{aligned}$$

According to the Gerschgorin disc theorem [8],  $\bar{A} < 0$ . By the Schur complement Lemma,  $\tilde{A} + \tilde{k}I < 0$ , which leads to the assertion of the theorem.  $\square$

Some simulations on the network (11) shown in Fig. 4 are presented in Fig. 5, where the dynamics of an isolated node, the inner liking matrix  $\Gamma$ , and the synchronized state are the same as that given in Section 2.

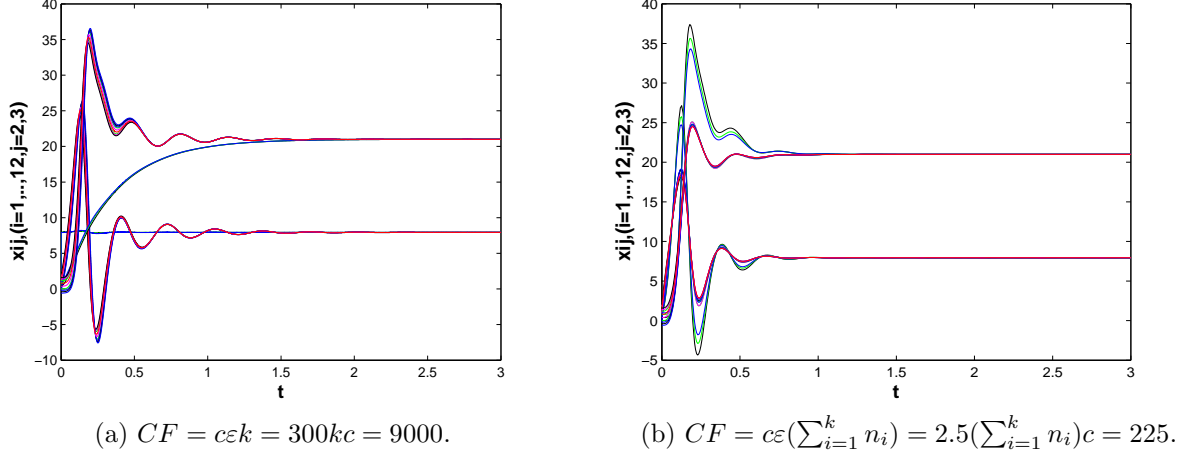


Fig. 5 Synchronization of network (11) with coupling strength  $c = 10$ : (a) pinning  $k$  central nodes; (b) pinning all the non-central nodes.

From Fig. 5, it can be seen that although all the non-central nodes are pinned, the cost is lower and the effect is better than the case where the central nodes are pinned.

By using the Kronecker product, the controlled network (5) can be rewritten as

$$\dot{e}(t) = F(x, s) + c(A - G) \otimes \Gamma e(t), \quad (13)$$

where  $e(t) = (e_1^T(t), \dots, e_N^T(t))^T$ ,  $F(x, s) = ((f(x_1(t)) - f(s(t)))^T, \dots, (f(x_N(t)) - f(s(t)))^T)^T$ ,  $G = \text{diag}\{\underbrace{0, \dots, 0}_k, \underbrace{\varepsilon, \dots, \varepsilon}_{N-k}\} \in R^{N \times N}$  and  $\Gamma = \text{diag}\{r_1, r_2, \dots, r_n\}$ ,  $r_i = 1$  or  $0$ ,  $i = 1, \dots, n$ .

**Theorem 3** If there are a positive diagonal matrix  $P$  and constants  $\mu > 0$ ,  $\varepsilon > 0$ ,  $\tilde{k} > 0$ ,  $c > 0$ , such that  $n_1 > \tilde{k}$ ,  $\varepsilon > \tilde{k} - 1$ ,  $\varepsilon > \frac{\tilde{k}(n_1+1-\tilde{k})}{n_1-\tilde{k}}$  and

$$(x - y)^T P(f(x, t) - f(y, t) - c\tilde{k}\Gamma(x - y)) \leq -\mu(x - y)^T(x - y),$$

then the controlled network (4) is globally exponentially synchronized to  $s(t)$ .

**Proof:** Choose a Lyapunov function as  $V(t) = \frac{1}{2}e^T(t)(I_N \otimes P)e(t)$ . Its time derivative is

$$\begin{aligned} \dot{V}(t) &= e^T(t)(I_N \otimes P)\dot{e}(t) \\ &= e^T(t)(I_N \otimes P)(F(x, s) + c(A - G) \otimes \Gamma e(t)) \\ &= e^T(t)(I_N \otimes P)F(x, s) + ce^T(t)((A - G) \otimes P\Gamma)e(t) \\ &= e^T(t)(I_N \otimes P)(F(x, s) - c(\tilde{k}I_n \otimes \Gamma)e^T(t)) + ce^T(t)((A - G) + \tilde{k}I_n) \otimes P\Gamma e(t). \end{aligned}$$

By Theorem 2,  $(A - G) + \tilde{k}I_n < 0$ . So,

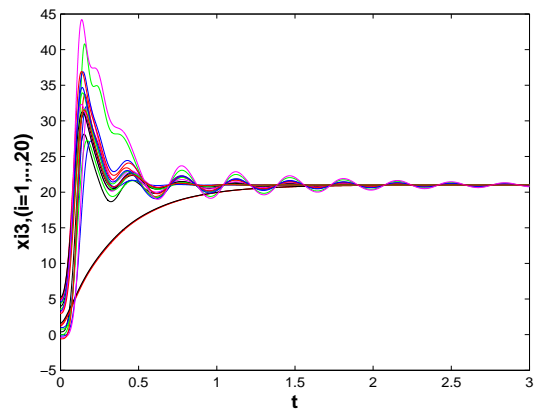
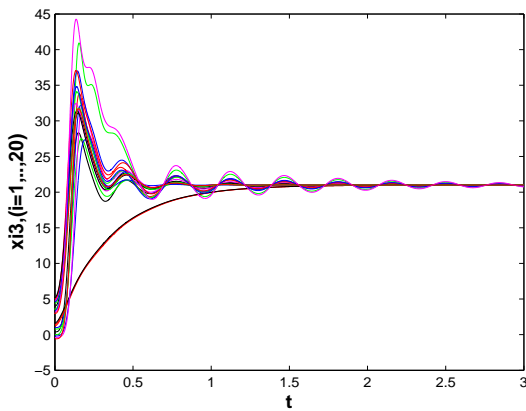
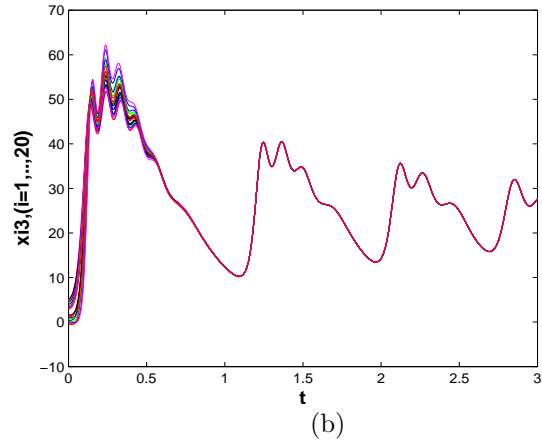
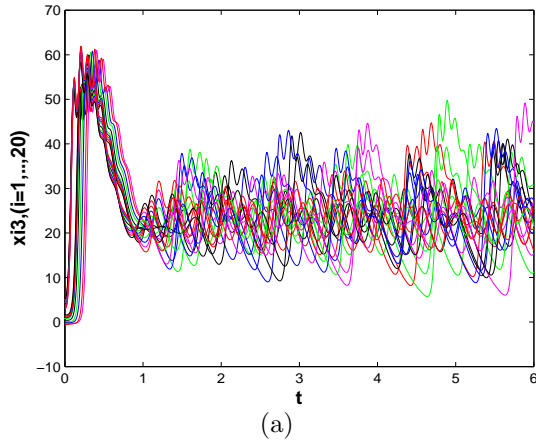
$$\begin{aligned} \dot{V}(t) &\leq -\mu \sum_{i=1}^N e_i^T(t)e_i(t) \leq -\mu \sum_{i=1}^N \frac{1}{\max\{P_j\}_{1 \leq j \leq n}} e_i^T(t)P e_i(t) \\ &= -\frac{\mu}{\max\{P_j\}_{1 \leq j \leq n}} e_i^T(t)(I_n \otimes P)e_i(t) = -\frac{\mu}{\max\{P_j\}_{1 \leq j \leq n}} V(t), \end{aligned}$$

and the theorem is thus proved.  $\square$

## 5 A network with multi-central nodes

## 6 Some non-regular complex networks

In this section, first, a non-regular coupled network consisting of 20 nodes is generated following the procedure of the well-known BA model. Fig. 8 shows the synchronization of the network with controllers being added to the three “biggest” nodes of degrees 15, 13 and 10, respectively. The dynamics of an isolated node, the inner liking matrix  $\Gamma$ , and the synchronized state are the same as that given in Section 2. From Figs. 8(c) and 8(d), it can be seen that the network does not synchronize any faster, although a larger feedback gain is used. In Fig. 9, a different control scheme is applied: eleven nodes with smaller degrees in the network are pinned with coupling strength  $c = 6$  and feedback gain  $\varepsilon = 5$ , yielding cost function  $CF = 330$ . Compared with Fig. 8, a better control performance is obtained with a much smaller cost function. Figs. 10 and 11 also show that it is more efficient by pinning nodes with smaller degrees than pinning nodes with larger degrees. As shown in Fig. 10, to achieve a similar synchronization effect, a much smaller feedback gain is needed in pinning “smaller” nodes than pinning “larger” ones, thereby the cost function is also smaller in the former control scheme than that in the latter. Fig. 11 shows that although the coupling strength, the feedback gain and the cost function are similar, the synchronization effect is better in pinning “smaller” nodes than pinning “larger” ones.



(c)

(d)

Fig. 8 Pinning the three “biggest” nodes with degrees 15, 13 and 10 in a 20-node non-regular coupled network: (a)  $c = 0$ ,  $\varepsilon = 0$ ,  $CF = 0$ . (b)  $c = 6$ ,  $\varepsilon = 0$ ,  $CF = 0$ . (c)  $c = 6$ ,  $\varepsilon = 500$ ,  $CF = 9000$ . (d)  $c = 6$ ,  $\varepsilon = 1000$ ,  $CF = 18000$ .

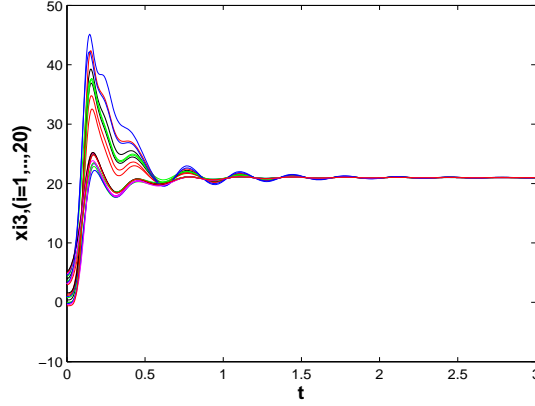
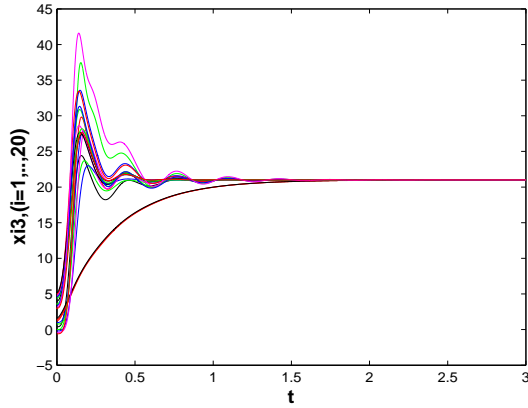
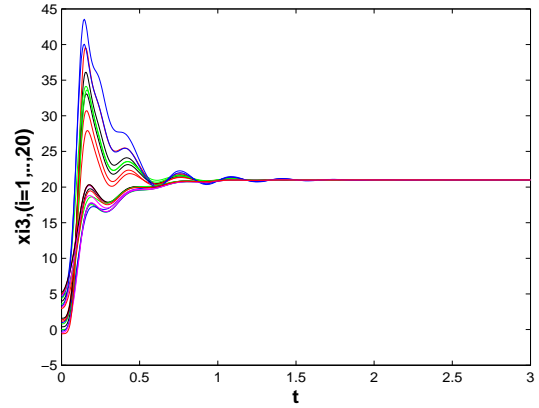


Fig. 9 Pinning the eleven “smaller” nodes in a 20-node non-regular coupled network:  $c = 6$ ,  $\varepsilon = 5$ ,  $CF = 330$ .

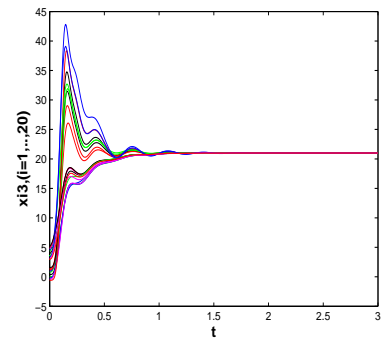
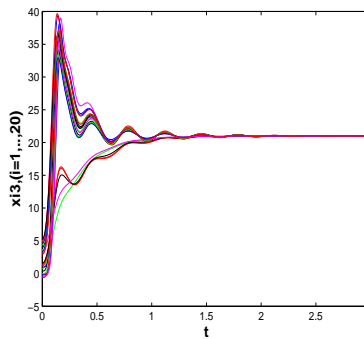
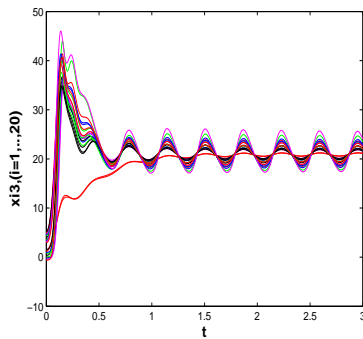


(a)  $c = 8$ ,  $\varepsilon = 500$ ,  $CF = 12000$ .



(b)  $c = 6$ ,  $\varepsilon = 8$ ,  $CF = 528$ .

Fig. 10 (a) Pinning the three “biggest” nodes with degrees 15, 13 and 10. (b) Pinning the eleven “smaller” nodes.



(a)  $c = 6$ ,  $\varepsilon = 55$ ,  $CF = 660$ .      (b)  $c = 6$ ,  $\varepsilon = 22$ ,  $CF = 660$ .      (c)  $c = 6$ ,  $\varepsilon = 10$ ,  $CF = 660$ .

Fig. 11 (a) Pinning the two “biggest” nodes with degrees 15 and 13. (b) Pinning the three “biggest” nodes with degrees 15, 13 and 10, and the two “smallest” nodes with the same degrees 3. (c) Pinning the eleven “smaller” nodes.

It seems more efficient to use the control scheme of pinning nodes with smaller degrees than pinning nodes with larger degrees for achieving the desired synchronous states of some non-regular coupled dynamical networks. This is an interesting phenomenon that was not noticed before.

## 7 Conclusions

In this paper, pinning control to achieve synchronization of complex dynamical networks is further investigated. In contrary to the general perception, the nodes with smaller degrees play an important role in the synchronizability of some networks. It has been shown by computer simulations in this paper that to achieve a similar synchronization effect on the networks considered, a smaller coupling strength, a smaller feedback gain and thus a lower cost function are needed in the control scheme of pinning nodes with smaller degrees comparing with those needed in pinning nodes with larger degrees. In other words, it seems more efficient to use the control scheme of pinning nodes with smaller degrees than that of pinning nodes with larger degrees for some dynamical networks to achieve synchronization, an interesting phenomenon that deserves further investigation in the future.

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